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Nonlinearity in Electro- and Magneto-statics with and without External Field

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Abstract

Due to the nonlinearity of QED, a static charge becomes a magnetic dipole if placed in a magnetic field. Already without external field, the cubic Maxwell equation for the field of a point charge has a soliton solution with a finite field energy. Equations are given for self-coupling dipole moments. Any theoretically found value for a multipole moment of a baryon or a meson should be subjected to nonlinear renormalization.

1 Introduction

In this talk we give an overview of the results presented in Refs.[1]–[5].

The nonlinear Maxwell equations truncated at the third power of their expansion in the expectation value of the electromagnetic field $a^\tau(x)$ over the blank vacuum or the vacuum with a background field have the form

$$j_\mu(x) + j_\mu^{\text{nl}}(x) = [\square\eta_{\mu\tau} - \partial_\mu\partial_\tau] a^\tau(x) + \int d^4y \Pi_{\mu\tau}(x, y) a^\tau(y) \quad (1)$$

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where $\square = \partial_0^2 - \nabla^2$. The four-vector $j_\mu(x)$ is the current to the field $a^\tau(x)$, whereas $j_\mu^{\text{nl}}(x)$ is the nonlinearly induced current, depending in its turn on the field:

$$j_\mu^{\text{nl}}(x) = -\frac{1}{2} \int d^4y d^4u \Pi_{\mu\tau\sigma}(x, y, u) a^\tau(y) a^\sigma(u) - \frac{1}{6} \int d^4y d^4u d^4v \Pi_{\mu\tau\sigma\rho}(x, y, u, v) a^\tau(y) a^\sigma(u) a^\rho(v)$$

The second-, third- and fourth-rank polarization tensors $\Pi_{\mu\tau}$, $\Pi_{\mu\tau\sigma}$ and $\Pi_{\mu\tau\sigma\rho}$ are responsible for the linear, quadratic and cubic responses of the vacuum to the applied electromagnetic field a^τ . These tensors are defined as the second, third and fourth variational derivatives of the effective action with respect to the fields. They depend on the background field if the latter is kept nonzero after these derivatives are calculated. When the field strength of the background is time- and space-independent, the polarization tensors of all ranks depend on the differences of the coordinates.

Nontrivial results concerning solutions of the nonlinear Maxwell equations (1, 2) may be already obtained provided we confine ourselves to the simplest approximation, which stems from the effective action $\Gamma = \int L(x) d^4x$ taken as a local functional, of the scalar (F) and pseudoscalar (G) field invariants, for instance exemplify it as the Heisenberg-Euler action in QED, or the Born-Infeld action beyond it. The local approximation is applicable to the fields, slowly varying in time and space. When QED is concerned, the corresponding space-time scale is determined by the electron Compton length m^{-1} . The truncation of the exact Maxwell equations at the third power of the field done above restricts their applicability to not too strong fields, but still stronger than the ones usually treated within the linear approximation $j_\mu^{\text{nl}} = 0$. In QED the measure of strength of the field is the value m^2/e . The whole approach is aimed to cover strong electromagnetic fields in the vicinity of elementary particles, although we cannot yet come too close to them owing to the above restrictions.

We shall consider here static fields of charges and of electric and magnetic dipoles. In Section 2 we shall deal with external constant magnetic field background and omit the cubic term in (2). In Section 3 there will be no background, hence $\Pi_{\mu\tau\sigma} = 0$ owing to the Furry theorem. So we shall face a cubic equation.

2 Magnetic moment of a spherical charge in a static and homogeneous magnetic background [1], [2], [4]

Here we disregard $\Pi_{\mu\tau\sigma\rho}$. The third-rank tensor is expressed in terms of the second and third derivatives L_{FF} , L_{GG} , L_{FGG} of the effective Lagrangian L with respect to the two electromagnetic field invariants taken at $G = 0$, $2F = B^2 = \text{const}$. They depend only on the background time- and space independent magnetic field $B = |\mathbf{B}|$. Then the current (2) in the totally static field-

configuration is [1]:

$$j_0^{\text{nl}}(\mathbf{x}) = 0, \quad j_i^{\text{nl}}(\mathbf{x}) = \epsilon_{ikj} \nabla_k \eta_j(\mathbf{x}), \quad (2)$$

$$\eta_i(\mathbf{x}) = \frac{B_i}{2} E^2 L_{FF} - \frac{B_i}{2} (\mathbf{B} \cdot \mathbf{E})^2 L_{FGG} - E_i (\mathbf{B} \cdot \mathbf{E}) L_{GG}. \quad (3)$$

Here \mathbf{E} is the static applied field contained in the vector-potential in the right-hand side of (2).

We treat here the nonlinearity as perturbation. Then E is understood as the electric field created by the static charge $j_0 \neq 0$, $j_i = 0$ via the equation (1), wherein the nonlinear current (2) is set equal to zero, $j_\mu^{\text{nl}} = 0$. Then the magnetic field produced by this electric field results from (2) and (3) via the Maxwell equations (1) without the second term in the r.-h. side (it would give a contribution of higher order; see, however, [4] for its full account) to be:

$$h_i(\mathbf{x}) = \left(\delta_{ik} - \frac{\nabla_i \nabla_k}{\nabla^2} \right) \eta_k(\mathbf{x}) = \eta_i(\mathbf{x}) + \frac{\partial_i \partial_k}{4\pi} \int d^3 y \frac{\eta_k(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \quad (4)$$

Consider the magnetic field, which is the response of the vacuum to the applied electric field, whose vector potential is chosen to be the following smooth central-symmetrical Coulomb-like function

$$a_0(r) = \left(-\frac{Ze}{8\pi R^3} r^2 + \frac{3Ze}{8\pi R} \right) \theta(R - r) + \frac{Ze}{4\pi r} \theta(r - R), \quad r = |\mathbf{x}|. \quad (5)$$

Here $\theta(z)$ is the step function, defined as

$$\theta(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0 \end{cases}. \quad (6)$$

Disregarding the higher-order effect of the linear electrization we state that (5) is the potential of the charge distributed inside the sphere with the radius R with the constant density $\rho(r) = \left(\frac{3}{4\pi} \frac{Ze}{R^3} \right) \theta(R - r)$. The long-range contribution of (4) calculated with the electric field contained in (5) (see [2] and [4] for its explicit form), $h_i^{\text{LR}}(\mathbf{x})$, behaves like a magnetic field generated by a magnetic dipole:

$$h_i^{\text{LR}}(\mathbf{x}) = \frac{3(\mathbf{x} \cdot \mathbf{M}) x_i}{r^5} - \frac{M_i}{r^3}, \quad (7)$$

with \mathbf{M} being the equivalent magnetic dipole moment, given by

$$M_i = \left(\frac{Ze}{4\pi} \right)^2 \frac{1}{5R} (3L_{FF} - 2L_{GG} - B^2 L_{FGG}) B_i. \quad (8)$$

The extension beyond the spherical symmetry may be found in [4].

3 Cubic self-interaction of electro- and magneto-static fields in blank vacuum [3], [5]

In this section no background field is present, hence $\Pi_{\mu\tau\sigma} = 0$ in (2). Besides, the principle of correspondence with the classical Faraday-Maxwell electromagnetism requires that also $\Pi_{\mu\tau} = 0$ within the local limit dealt with here, because in this limit the quantum theory is normalized to the classical one. Consequently, there is no linear polarization, and inductions and field strengths are the same in the blank vacuum.

Unlike the previous section, now the nonlinearity in the Maxwell equation (1) will not be taken as small, but treated seriously. In the two subsections below we include only the static cases, where, besides, the field a_μ in the nonlinear current (2) carries either only electric, E , or only magnetic, B , field, not the both simultaneously. Then this current is calculated to be

$$j_0^{\text{nl}}(x) = \frac{1}{2} L_{FF} \partial_i [(B^2 - E^2) E_i], \quad j_i^{\text{nl}}(x) = -\frac{1}{2} L_{FF} \partial_j [(B^2 - E^2) B_k] \epsilon_{ijk}. \quad (9)$$

In the present section the derivative $L_{FF} \equiv \gamma$ is understood as taken at $F = G = 0$.

3.1 Self-coupling of a charge. Finiteness of the point-charge electrostatic field-energy

Let there be a point charge e placed at the origin $r = 0$. We are looking for a spherically symmetric solution of the Maxwell equation (1) with $B = 0$, which, given the nonlinear current (9), takes the form (we denote $\gamma \equiv L_{FF}$ for brevity)

$$\nabla \left[\left(1 + \frac{\gamma}{2} E^2 \right) \mathbf{E} \right] = 0, \quad (10)$$

valid everywhere outside the origin $\mathbf{x} = 0$, since $j_0 = 0$ there. At large r the standard Coulomb field of the point charge e

$$\frac{e}{4\pi r^2} \frac{\mathbf{x}}{r}, \quad (11)$$

should be implied as the boundary condition. Then with the spherically symmetric Ansatz $E(r) \frac{\mathbf{x}}{r} = \mathbf{E}(\mathbf{x})$ equation (10) is solved as

$$\left(1 + \frac{\gamma}{2} E^2(r) \right) E(r) = \frac{e}{4\pi r^2}. \quad (12)$$

This cubic equation is readily solved by the Cardan formula (see [5] for the explicit representation), but the most important thing about its solution is clear without solving it: at short distances

$r \rightarrow \infty$ the field E also infinitely grows, hence one can neglect the unity in (12) to immediately obtain

$$E(r) \sim \left(\frac{e}{2\pi\gamma}\right)^{\frac{1}{3}} \left(\frac{1}{r}\right)^{\frac{2}{3}}, \quad (13)$$

The behavior (13) of the electrostatic field, produced by the point charge e via the nonlinear field equations (1), is essentially less singular in the vicinity of the charge than the standard Coulomb field $\frac{e}{4\pi r^2}$.

Let us see that this suppression of the singularity is enough to provide convergence of the integrals giving the energy of the field configuration that solves equation (10). To this end note that on the subclass of electromagnetic field we are considering here, the equations of motion (1), or, equivalently, (10) are generated by the quartic Lagrangian

$$-F(x) + L = -F(x) + \frac{\gamma}{2} (F(x))^2. \quad (14)$$

With this Lagrangian, the energy density calculated on spherically-symmetric electric field configuration following the Noether theorem is

$$\Theta^{00} = \frac{E^2}{2} + \frac{3\gamma E^4}{8}. \quad (15)$$

The behaviour (13) provides the ultraviolet, near $|\mathbf{x}| = 0$, convergence of the electrostatic field energy $\int \Theta^{00} d^3x$ of the point charge. As for the convergence of this integral at $|\mathbf{x}| \rightarrow \infty$, it is provided by the standard long-range Coulomb behaviour (11) of the solution to equation (10) obtained by neglecting the second term inside the bracket as compared to the unity in the far-off region.

The explicit use of the Cardan formula in (15) allows to calculate the integral for the field energy. If the value

$$L_{FF} = \frac{e^4}{45\pi^2 m^4}, \quad (16)$$

where e and m are the electron charge and mass, is calculated referring to the Euler-Heisenberg effective Lagrangian of QED and substituted for γ , the result for the "rest mass of the electron," understood as a point charge, is about twice the true electron mass:

$$\int \Theta^{00} d^3x = 2.09m. \quad (17)$$

The conclusion about finiteness of the electrostatic field energy of a point charge is readily extended [5] to the nonlinear electrodynamics with the effective Lagrangian being any-power polynomial of the field invariants, thereby also to QED truncated at any finite term of its Taylor

expansion in powers of the field in place of (1).

3.2 Self-coupling of magnetic and electric dipoles

Consider first a magnetic dipole. This means that only B is kept in the nonlinear current (9), hence $j_0^{nl} = 0$. As for the nonlinear 3-current, it is expressed as

$$j_i^{nl}(\mathbf{x}) = \epsilon_{ijk} \nabla_j \eta_k(\mathbf{x}), \quad \eta_i(\mathbf{x}) = -\frac{1}{2} L_{FF} B_i(\mathbf{x}) B^2(\mathbf{x}) \quad (18)$$

in terms of the auxiliary magnetic field η analogous to (3). Eq. (4) is again valid for the magnetic field induced by the nonlinear current, this time without the reservations made in the previous section about the disregard of the linear magnetization. This field is to be added to the initial magnetic field \mathbf{h}^{nl} (linearly produced by the current \mathbf{j}) to make the total resulting magnetic field $\mathbf{h}^{tot} = \mathbf{h}^{nl} + \mathbf{h}$.

Let there be a sphere with the radius R , and a time-independent current $\mathbf{j}(\mathbf{x})$ concentrated on its surface:

$$\mathbf{j}(\mathbf{x}) = \frac{\mathbf{M}^{(0)} \times \mathbf{x}}{r^4} \delta(r - R). \quad (19)$$

Here $\mathbf{M}^{(0)}$ is a constant vector directed, say, along the axis 3. The current density (19) obeys the continuity condition $\nabla \cdot \mathbf{j} = 0$, its flow lines are circular in the planes parallel to the plane (1,2). The magnetic field produced by this current via the Maxwell equation $\nabla \times \mathbf{h}^{lin}(\mathbf{x}) = \mathbf{j}(\mathbf{x})$ is

$$\mathbf{h}^{lin}(\mathbf{x}) = \theta(R - r) \frac{2\mathbf{M}^{(0)}}{R^3} + \theta(r - R) \left(-\frac{\mathbf{M}^{(0)}}{r^3} + 3\frac{\mathbf{x} \cdot \mathbf{M}^{(0)}}{r^5} \mathbf{x} \right). \quad (20)$$

Outside the sphere this is the magnetic dipole field with the constant vector density $\mathbf{M}^{(0)}$ playing the role of its magnetic moment. Using this expression in the r.h. side of Eq. (4), after a lengthy calculation the nonlinear correction h to the field (20) of the magnetic dipole (19) was obtained in [3] both inside and outside the sphere. At large distances the resulting field reproduces the original magnetic dipole behaviour:

$$\mathbf{h}^{tot}(\mathbf{r})|_{r \gg R} = \mathbf{h}^{lin}(\mathbf{r}) \left(1 - \frac{7}{5} L_{FF} \frac{M^{(0)2}}{R^6} \right). \quad (21)$$

Once we want to treat the nonlinearity seriously, and not just as a perturbation, we should for self-consistency demand that the magnetic field forming the nonlinear current (18) be not (20), but its final result, which is again the magnetic dipole field, but with the bare magnetic moment $\mathbf{M}^{(0)}$ replaced by the final magnetic moment to be denoted as \mathbf{M} . Then in the long range for the

total field we obtain

$$-\frac{\mathbf{M}}{r^3} + 3\frac{\mathbf{x} \cdot \mathbf{M}}{r^5}\mathbf{x} = -\frac{\mathbf{M}^{(0)}}{r^3} + 3\frac{\mathbf{x} \cdot \mathbf{M}^{(0)}}{r^5}\mathbf{x} - \left(\frac{\mathbf{M}}{r^3} + 3\frac{\mathbf{x} \cdot \mathbf{M}}{r^5}\mathbf{x}\right) \left(\frac{7}{5}L_{FF}\frac{M^2}{R^6}\right). \quad (22)$$

From this the equation for self-coupling of the magnetic moment follows to be:

$$\mathbf{M} \left(1 + \frac{7}{5}L_{FF}\frac{M^2}{R^6}\right) = \mathbf{M}^{(0)} \quad (23)$$

Analogous equation for the electric moment is

$$\mathbf{p} \left(1 + \frac{1}{10}L_{FF}\frac{p^2}{R^6}\right) = \mathbf{p}^{(0)}. \quad (24)$$

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